

Regularity of some invariant distributions on nice symmetric pairs

Pascale Harinck*

Abstract

J. Sekiguchi determined the semisimple symmetric pairs $(\mathfrak{g}, \mathfrak{h})$, called nice symmetric pairs, on which there is no non-zero invariant eigendistribution with singular support. On such pairs, we study regularity of invariant distributions annihilated by a polynomial of the Casimir operator. We deduce that invariant eigendistributions on $(\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))$ are locally integrable functions.

Mathematics Subject Classification 2000: MSC classification 22E30

Keywords and phrases: Nice symmetric pairs, invariant distributions, eigendistributions, transfer of distributions, radial part of differential operators.

Introduction

Let G be a reductive group such that $\mathrm{Ad}(G)$ is connected. Let σ be an involutive automorphism of G . We denote by the same letter σ the corresponding involution on the Lie algebra \mathfrak{g} of G . Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition into $+1$ and -1 eigenspaces with respect to σ . Then $(\mathfrak{g}, \mathfrak{h})$ is called a reductive symmetric pair (or semisimple when \mathfrak{g} is semisimple). Let H be the group of fixed points of σ in G .

In [7], J. Sekiguchi describes semisimple symmetric pairs on which there is no non-zero invariant eigendistribution with support in $\mathfrak{q} - \mathfrak{q}^{reg}$ where \mathfrak{q}^{reg} is the set of semisimple regular elements of \mathfrak{q} . These pairs, called nice symmetric pairs, are characterized by a property on distinguished nilpotent elements and we can generalize this notion to reductive pairs (Definition 4.1). Our main result is the following. Let ω be the Casimir polynomial of \mathfrak{q} and $\partial(\omega)$ the corresponding differential operator on \mathfrak{q} .

Theorem 0.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a nice reductive symmetric pair. Let \mathcal{V} be an H -invariant open subset of \mathfrak{q} . Let Θ be an H -invariant distribution on \mathcal{V} such that*

1. *There exists $P \in \mathbb{C}[X]$ such that $P(\partial(\omega))\Theta = 0$,*
2. *There exists $F \in L^1_{loc}(\mathcal{V})^H$ such that $\Theta = F$ on $\mathcal{V} \cap \mathfrak{q}^{reg}$.*

Then $\Theta = F$ as distribution on \mathcal{V} .

In [2], E. Galina and Y. Laurent obtained stronger results on invariant distributions on nice symmetric pairs by different methods based on algebraic properties of \mathcal{D} -modules. They proved

*Ecole Polytechnique, CMLS - CNRS UMR 7640, Route de Saclay 91128 Palaiseau Cédex, harinck@math.polytechnique.fr

that any invariant distribution on a nice pair which is annihilated by a finite codimensional ideal of the algebra of H -invariant differential operators with constant coefficients on \mathfrak{q} is a locally integrable function ([2] Corollary 1.7.6).

Our approach uses properties of distributions. Assuming that $S = \Theta - F$ is non-zero, we are led to a contradiction. By the work of G. van Dijk ([8]) and J. Sekiguchi ([7]), we can adapt the descent method of Harish-Chandra. Thus, we construct a non-zero distribution \tilde{S} defined on a neighborhood W of 0 in $\mathbb{R}^r \times \mathbb{R}^m$ with support in $(\{0\} \times \mathbb{R}^m) \cap W$ such that there exist a locally integrable function \tilde{F} on W and a differential operator D , which is obtained from radial parts of $\partial(\omega)$ near semisimple elements and nilpotent elements, satisfying $P(D)\tilde{S} = P(D)\tilde{F}$. Using the method developed by M. Atiyah in [1], one studies the degree of singularity along $\{0\} \times \mathbb{R}^m$ of different distributions in this equation. One deduces that $\tilde{S} = 0$ and thus a contradiction.

In the last section, we complete the results of [3] on the nice symmetric pair $(\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))$ and deduce that any invariant eigendistribution for a regular character on this pair is given by a locally integrable function.

1 Notation

Let M be a smooth variety. Let $C^\infty(M)$ be the space of smooth functions on M , $\mathcal{D}(M)$ the subspace of compactly supported smooth functions, $L_{loc}^1(M)$ the space of locally integrable functions on M , endowed with their standard topology and $\mathcal{D}'(M)$ the space of distributions on M .

For a group G acting on M , one denotes by \mathcal{F}^G the points of \mathcal{F} fixed by G for each space \mathcal{F} defined as above.

If $N \subset M$ and if f is a function defined on M , one denotes by $f|_N$ its restriction to N .

If V is a finite dimensional real vector space then V^* is its algebraic dual and $V_{\mathbb{C}}$ is its complexified vector space. The symmetric algebra $S[V]$ of V can be identified to the space $\mathbb{R}[V^*]$ of polynomial functions on V^* with real coefficients and to the space of differential operators with real constant coefficients on V . Similarly, one has $S[V_{\mathbb{C}}] = \mathbb{C}[V^*]$ and this algebra can be identified to the space of differential operators with complex constant coefficients on $V_{\mathbb{C}}$. If $u \in S[V]$ (resp. $S[V_{\mathbb{C}}]$), then $\partial(u)$ will denote the corresponding differential operator.

Let G be a reductive group such that $\text{Ad}(G)$ is connected, and σ an involution on G . This defines an involution, denoted by the same letter σ on the Lie algebra \mathfrak{g} of G . Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the direct decomposition of \mathfrak{g} into the $+1$ and -1 eigenspaces of σ . Then $(\mathfrak{g}, \mathfrak{h})$ is called a reductive symmetric pair. Let H be the subgroup of fixed points of σ in G .

Let $\mathfrak{c}_{\mathfrak{g}}$ be the center of \mathfrak{g} and \mathfrak{g}_s its derived algebra. We set

$$\mathfrak{c}_{\mathfrak{q}} = \mathfrak{c}_{\mathfrak{g}} \cap \mathfrak{q} \text{ and } \mathfrak{q}_s = \mathfrak{g}_s \cap \mathfrak{q}.$$

If x is an element of \mathfrak{g} and \mathfrak{r} is a subspace of \mathfrak{g} , we denote by \mathfrak{r}_x the centralizer of x in \mathfrak{r} .

We fix a non-degenerate bilinear form B on \mathfrak{g} which is equal to the Killing form on \mathfrak{g}_s . Then $\omega(X) = B(X, X)$ is the Casimir polynomial of \mathfrak{q} .

2 Transfer of distributions and differential operators

We recall results of ([8] sections 2 and 3) and ([7] section (3.2)) on restriction of distributions and radial parts of differential operators. Their proofs are similar to ([4] or [10] Part I, chapter 2).

Let $x_0 \in \mathfrak{q}_s$. Let U be a linear subspace of \mathfrak{q} such that $\mathfrak{q} = U \oplus [x_0, \mathfrak{h}]$ and V be a linear subspace of \mathfrak{h} such that $\mathfrak{h} = V \oplus \mathfrak{h}_{x_0}$. Consider the open subset ${}^{\vee}U = \{Z \in U; U + [x_0 + Z, \mathfrak{h}] = \mathfrak{q}\}$ containing 0. Then the map Ψ from $H \times {}^{\vee}U$ to \mathfrak{q} defined by $\Psi(h, u) = h \cdot (x_0 + u)$ is a submersion. In particular, $\Omega = \Psi(H \times {}^{\vee}U)$ is an open H -invariant subset of \mathfrak{q} containing x_0 . We fix an Haar measure dh on H and we denote by du (respectively dx) the Lebesgue measure on U (respectively \mathfrak{q}). The submersion Ψ induces a continuous surjective map Ψ_* from $\mathcal{D}(H \times {}^{\vee}U)$ onto $\mathcal{D}(\Omega)$ such that, for any $F \in L^1_{loc}(\mathfrak{q})$ and any $f \in \mathcal{D}(H \times {}^{\vee}U)$, one has

$$\int_{H \times U} F \circ \Psi(h, u) f(h, u) dh du = \int_{\mathfrak{q}} F(x) \Psi_*(f)(x) dx.$$

Theorem 2.1. *For $T \in \mathcal{D}'(\Omega)^H$ there exists a unique distribution $\mathcal{R}es_U T$ defined on ${}^{\vee}U$, called the restriction of T to ${}^{\vee}U$ with respect to Ψ , such that for any $f \in \mathcal{D}(H \times {}^{\vee}U)$, one has*

$$\langle T, \Psi_*(f) \rangle = \langle \mathcal{R}es_U T, p_*(f) \rangle$$

where $p_*(f) \in \mathcal{D}(U)$ is defined by $p_*(f)(u) = \int_H f(h, u) dh$.

This restriction satisfies the following properties:

1. If U is stable under the action of a subgroup H_0 of H then $\mathcal{R}es_U T$ is H_0 -invariant.
2. $x_0 + \text{supp}(\mathcal{R}es_U T) \subset \text{supp}(T) \cap (x_0 + {}^{\vee}U)$.
3. If $F \in L^1_{loc}(\Omega)^H$ then $\mathcal{R}es_U F$ is the locally integrable function on ${}^{\vee}U$ defined by $\mathcal{R}es_U F(u) = F(x_0 + u)$.
4. If $\mathcal{R}es_U T = 0$ then $T = 0$ on Ω .

Theorem 2.2. *Let D be a H -invariant differential operator on \mathfrak{q} . Then there exists a differential operator $\mathcal{R}ad_U(D)$, called the radial part of D with respect to Ψ , defined on ${}^{\vee}U$ such that for any $f \in \mathcal{D}(\Omega)^H$, one has $(D \cdot f)(x_0 + u) = \mathcal{R}ad_U(D) \cdot \mathcal{R}es_U f(u)$ for $u \in {}^{\vee}U$.*

Moreover, for any $T \in \mathcal{D}'(\Omega)^H$, one has

$$\mathcal{R}es_U(D \cdot T) = \mathcal{R}ad_U(D) \cdot \mathcal{R}es_U(T).$$

3 Semisimple elements

We recall that a Cartan subspace of \mathfrak{q} is a maximal abelian subspace of \mathfrak{q} consisting of semisimple elements.

If $\mathfrak{r} = \mathfrak{q}$ or \mathfrak{q}_s , we denote by $\mathcal{S}(\mathfrak{r})$ the set of semisimple elements of \mathfrak{r} .

Let \mathfrak{a} be a Cartan subspace of \mathfrak{q} . If $\lambda \in \mathfrak{g}_{\mathbb{C}}^*$, we set

$$\mathfrak{g}_{\mathbb{C}}^{\lambda} = \{X \in \mathfrak{g}_{\mathbb{C}}; [A, X] = \lambda(A)X \text{ for any } A \in \mathfrak{a}_{\mathbb{C}}\}$$

and

$$\Sigma(\mathfrak{a}) = \{\lambda \in \mathfrak{g}_{\mathbb{C}}^*; \mathfrak{g}_{\mathbb{C}}^{\lambda} \neq \{0\}\}.$$

Then $\Sigma(\mathfrak{a})$ is the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$.

An element X of $\mathcal{S}(\mathfrak{q})$ is \mathfrak{q} -regular (or regular) if its centralizer \mathfrak{q}_X in \mathfrak{q} is a Cartan subspace. If $X \in \mathfrak{a}$ then X is regular if and only if $\lambda(X) \neq 0$ for all $\lambda \in \Sigma(\mathfrak{a})$. We denote by \mathfrak{q}^{reg} the open dense subset of semisimple regular elements of \mathfrak{q} .

Let $A_0 \in \mathcal{S}(\mathfrak{q})$. Its centralizer $\mathfrak{z} = \mathfrak{g}_{A_0}$ in \mathfrak{g} is a reductive σ -stable Lie subalgebra of \mathfrak{g} . We denote by \mathfrak{c} its center and by \mathfrak{z}_s its derived algebra. We set

$$\mathfrak{c}^- = \mathfrak{c} \cap \mathfrak{q}, \quad \mathfrak{c}^+ = \mathfrak{c} \cap \mathfrak{h}, \quad \mathfrak{z}_s^- = \mathfrak{z}_s \cap \mathfrak{q} \quad \text{and} \quad \mathfrak{z}_s^+ = \mathfrak{z}_s \cap \mathfrak{h}.$$

The pair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ is a semisimple symmetric subpair of $(\mathfrak{g}_s, \mathfrak{h}_s)$ which is equal to $(\mathfrak{g}_s, \mathfrak{h}_s)$ if $A_0 \in \mathfrak{c}_{\mathfrak{q}}$. Let H_s^+ be the analytic subgroup of H with Lie algebra \mathfrak{z}_s^+ .

We assume that $A_0 \notin \mathfrak{c}_{\mathfrak{q}}$. We take a Cartan subspace \mathfrak{a} of \mathfrak{q} containing A_0 and consider the corresponding root system $\Sigma = \Sigma(\mathfrak{a})$. We fix a positive system Σ^+ of Σ . For any $\lambda \in \Sigma^+$, we choose a \mathbb{C} -basis $X_{\lambda,1}, \dots, X_{\lambda,m_{\lambda}}$ of $\mathfrak{g}_{\mathbb{C}}^{\lambda}$ such that $B(X_{\lambda,i}, \sigma(X_{\lambda,j})) = -\delta_{i,j}$ for $i, j \in \{1, \dots, m_{\lambda}\}$. Let $\Sigma_1^+ = \{\lambda \in \Sigma^+; \lambda(A_0) \neq 0\}$. We set

$$V_{\mathbb{C}}^{\pm} = \sum_{\lambda \in \Sigma_1^+} \sum_{j=1}^{m_{\lambda}} (X_{\lambda,j} \pm \sigma(X_{\lambda,j})), \quad V^+ = V_{\mathbb{C}}^+ \cap \mathfrak{h}, \quad V^- = V_{\mathbb{C}}^- \cap \mathfrak{q}.$$

We have the decompositions $\mathfrak{h} = \mathfrak{z}^+ \oplus V^+$ and $\mathfrak{q} = \mathfrak{z}^- \oplus V^-$, with $\dim V^+ = \dim V^-$ and $[A_0, \mathfrak{h}] = V^-$.

If $Z_0 \in \mathfrak{z}^-$, we define the map η_{Z_0} from $V^+ \times \mathfrak{z}^-$ to \mathfrak{q} by $\eta_{Z_0}(v, Z) = Z + [v, A_0 + Z_0]$. Then η_0 is a bijective map. We set $\xi(Z_0) = \det(\eta_{Z_0} \circ \eta_0^{-1})$ and $\mathfrak{z}^- = \{Z \in \mathfrak{z}^-; \xi(Z) \neq 0\}$. Then \mathfrak{z}^- is invariant under H_s^+ .

Thus the map γ from $H \times \mathfrak{z}^-$ to \mathfrak{q} defined by $\gamma(h, Z) = h \cdot (A_0 + Z)$ is a submersion. By Theorem 2.1, for any H -invariant distribution Θ on \mathfrak{q} , there exists a unique H_s^+ -invariant distribution $\mathcal{R}es_{\mathfrak{z}^-} \Theta$ defined on \mathfrak{z}^- such that, for any $f \in \mathcal{D}(H \times \mathfrak{z}^-)$, one has $\langle \Theta, \gamma_*(f) \rangle = \langle \mathcal{R}es_{\mathfrak{z}^-} \Theta, p_*(f) \rangle$.

Let $\omega_{\mathfrak{z}^-}$ be the restriction of ω to \mathfrak{z}^- . Then, one has:

Lemma 3.1. ([7]) *Lemma 4.4). Let $\mathcal{R}ad_{\mathfrak{z}^-}(\partial(\omega))$ be the radial part of $\partial(\omega)$ with respect to γ (Theorem 2.2). Then*

$$\mathcal{R}ad_{\mathfrak{z}^-}(\partial(\omega)) = \xi^{-1/2} \partial(\omega_{\mathfrak{z}^-}) \circ \xi^{1/2} - \mu$$

where $\mu(Z) = \xi(Z)^{-1/2} (\partial(\omega_{\mathfrak{z}^-}) \xi^{1/2})(Z)$ is an analytic function on \mathfrak{z}^- .

4 Nilpotent and distinguished elements

Let $Z_0 \in \mathfrak{q}$. Let $Z_0 = A_0 + X_0$ be its Jordan decomposition ([7] Lemma 1.1). We construct the symmetric pair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ related to A_0 as in 3.

We assume that X_0 is different from zero. From ([7] Lemma 1.7), there exists a normal sl_2 -triple (B_0, X_0, Y_0) of $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ containing X_0 , i.e. satisfying $B_0 \in \mathfrak{z}_s^+$ and $Y_0 \in \mathfrak{z}_s^-$ such that $[B_0, X_0] = 2X_0$, $[B_0, Y_0] = -2Y_0$ and $[X_0, Y_0] = B_0$.

We set $\mathfrak{z}_0 = \mathbb{R}B_0 + \mathbb{R}X_0 + \mathbb{R}Y_0$. The Cartan involution θ_0 of \mathfrak{z}_0 defined by $\theta_0 : (B_0, X_0, Y_0) \rightarrow (-B_0, -Y_0, -X_0)$ extends to a Cartan involution of \mathfrak{z}_s , denoted by θ , which commutes with σ . ([8] Lemma 1). The bilinear form $(X, Y) \mapsto -B(\theta(X), Y)$ defines a scalar product on \mathfrak{z}_s .

We can decompose \mathfrak{z}_s in an orthogonal sum $\mathfrak{z}_s = \sum_i \mathfrak{z}_i$ of irreducible representations \mathfrak{z}_i under the adjoint action of \mathfrak{z}_0 . One can choose a suitable ordering of the \mathfrak{z}_i such that $(\mathfrak{z}_s^-)_{Y_0} = \sum_{i=1}^r \mathfrak{z}_i \cap (\mathfrak{z}_s^-)_{Y_0} = \theta((\mathfrak{z}_s^-)_{X_0})$ with $\mathfrak{z}_1 = \mathfrak{z}_0$ and $\dim \mathfrak{z}_i \cap (\mathfrak{z}_s^-)_{Y_0} = 1$. We set $n_i + 1 = \dim \mathfrak{z}_i$. Hence, there exists an orthonormal basis (w_1, \dots, w_r) of $(\mathfrak{z}_s^-)_{Y_0}$ such that $w_1 = \frac{Y_0}{\|Y_0\|}$ and $[B_0, w_i] = -n_i w_i$ for $i \in \{1, \dots, r\}$. In particular, one has $n_1 = 2$.

We set

$$\delta_{\mathfrak{q}}(Z_0) = \delta_{\mathfrak{z}_s^-}(X_0) = \sum_{i=1}^r (n_i + 2) - \dim(\mathfrak{z}_s^-).$$

Let $\mathcal{N}(\mathfrak{z}_s^-)$ be the set of nilpotent elements of \mathfrak{z}_s^- .

Definition 4.1. ([7] Definitions 1.11 and 1.13)

1. An element X_0 of $\mathcal{N}(\mathfrak{z}_s^-)$ is a \mathfrak{z}_s^- -distinguished nilpotent element if $(\mathfrak{z}_s^-)_{X_0}$ contains no non-zero semisimple element.
2. An element Z_0 of \mathfrak{q} with Jordan decomposition $Z_0 = A_0 + X_0$ is called \mathfrak{q} -distinguished if X_0 is a \mathfrak{z}_s^- -distinguished nilpotent element of \mathfrak{z}_s^- .

Definition 4.2. The symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is nice if for any \mathfrak{q} -distinguished element Z , one has $\delta_{\mathfrak{q}}(Z) > 0$.

Let ω_s be the restriction of ω to \mathfrak{z}_s^- . Though ω_s is not the Casimir polynomial on \mathfrak{z}_s^- , one has the following result:

Lemma 4.3. ([8] Lemma 4) The following assertions are equivalent:

1. X_0 is a \mathfrak{z}_s^- -distinguished nilpotent element.
2. $\omega_s(X) = 0$ for all $X \in (\mathfrak{z}_s^-)_{X_0}$.
3. $\omega_s(X) = 0$ for all $X \in (\mathfrak{z}_s^-)_{Y_0}$.
4. $n_i > 0$.
5. $(\mathfrak{z}_s^-)_{X_0} \cap (\mathfrak{z}_s^-)_{Y_0} = \{0\}$.

Thus, if X_0 is a \mathfrak{z}_s^- -distinguished nilpotent element then one has $\omega(X_0 + X) = 2B(X_0, X) = 2\|Y_0\|x_1$ for all $X \in (\mathfrak{z}_s^-)_{Y_0}$, where x_1 is the first coordinate of X in the basis (w_1, \dots, w_r) of $(\mathfrak{z}_s^-)_{Y_0}$.

For any $X_0 \in \mathcal{N}(\mathfrak{z}_s^-)$, one has $\mathfrak{z}_s^- = (\mathfrak{z}_s^-)_{Y_0} \oplus [\mathfrak{z}_s^+, X_0]$ and $\mathfrak{z}_s^+ = (\mathfrak{z}_s^+)_{X_0} \oplus [\mathfrak{z}_s^-, Y_0]$. From now on, we set

$$U = (\mathfrak{z}_s^-)_{Y_0}.$$

For $X \in U$, we consider the map ψ_X from $[\mathfrak{z}_s^-, Y_0] \times U$ to \mathfrak{z}_s^- defined by $\psi_X(v, z) = z + [v, X_0 + X]$. The map ψ_0 is bijective.

We set $\kappa(X) = \det(\psi_X \circ \psi_0^{-1})$ and $U = \{X \in U; \kappa(X) \neq 0\}$. Hence, the map π from $H_s^+ \times U$ to \mathfrak{z}_s^- defined by $\pi(h, X) = h \cdot (X_0 + X)$ is a submersion.

We precise now some properties of π related to $\mathcal{N}(\mathfrak{z}_s^-)$.

By ([9] Theorem 23]), we can write $\mathcal{N}(\mathfrak{z}_s^-) = \mathcal{O}_1 \cup \dots \cup \mathcal{O}_\nu$ where the \mathcal{O}_j are disjoint H_s^+ -orbits with $\mathcal{O}_\nu = \{0\}$ and each \mathcal{O}_j is open in the closed set $\mathcal{N}_j = \mathcal{O}_j \cup \dots \cup \mathcal{O}_\nu$. One assumes that $\mathcal{O}_j = H_s^+ \cdot X_0$.

Lemma 4.4. ([8] Lemma 17 and 18). *There exists a neighborhood U_0 of 0 in U such that*

1. π is a submersion on $H_s^+ \times U_0$,
2. $\Omega_0 = \pi(H_s^+ \times U_0)$ is an open neighborhood of X_0 in \mathfrak{z}_s^- and $\Omega_0 \cap \mathcal{N}_j = \mathcal{O}_j$,
3. $\mathcal{O}_j \cap (X_0 + U_0) = \{X_0\}$
4. Let Θ be an H_s^+ -invariant distribution on Ω_0 . Let $\mathcal{R}es_U \Theta$ be its restriction to U with respect to π .
If $\text{supp}(\Theta) \subset \mathcal{N}_j$ then $\text{supp}(\mathcal{R}es_U \Theta) \subset \{0\}$.

We denote by $\omega_{\mathfrak{c}^-}$ and ω_s the restrictions of ω to \mathfrak{c}^- and \mathfrak{z}_s^- respectively. One has $\omega_{\mathfrak{z}_s^-} = \omega_{\mathfrak{c}^-} + \omega_s$. We precise now the radial part $\mathcal{R}ad_U(\partial(\omega_s))$ of $\partial(\omega_s)$ with respect to π . We denote by $\mathcal{R}ad_{U,X}(\partial(\omega_s))$ its local expression at $X \in U_0$.

Lemma 4.5. ([8] Lemma 13) *The homogeneous part of degree 2 of $\mathcal{R}ad_{U,0}(\partial(\omega_s))$ is zero if and only if X_0 is \mathfrak{z}_s^- -distinguished.*

Theorem 4.6. ([8] Theorem 14) *Let X_0 be a \mathfrak{z}_s^- -distinguished nilpotent element and $c_0 = \|X_0\|$. Then, there exist analytic functions $a_{i,j}$ ($2 \leq i, j \leq r$) and a_i ($2 \leq i \leq r$) on U_0 satisfying $a_{i,j}(0) = 0$ such that, for any H_s^+ -invariant distribution T on Ω_0 , one has*

$$\begin{aligned} \mathcal{R}es_U(\partial(\omega_s)T) &= \mathcal{R}ad_U((\partial(\omega_s))\mathcal{R}es_U(T)) \\ &= \frac{1}{c_0} \left(2x_1 \frac{\partial^2}{\partial x_1^2} + (\dim \mathfrak{z}_s^-) \frac{\partial}{\partial x_1} + \sum_{i=2}^r (n_i + 2)x_i \frac{\partial^2}{\partial x_1 \partial x_i} \right. \\ &\quad \left. + \sum_{2 \leq i \leq j \leq r} a_{i,j}(X) \frac{\partial^2}{\partial x_j \partial x_i} + \sum_{i=2}^r a_i(X) \frac{\partial}{\partial x_i} \right) \mathcal{R}es_U(T) \end{aligned}$$

where x_1, \dots, x_r are the coordinates of X in the basis (w_1, \dots, w_r) .

5 The main Theorem

Our goal is to prove the following Theorem:

Theorem 5.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a nice reductive symmetric pair. Let \mathcal{V} an H -invariant open subset of \mathfrak{q} . Let Θ be an H -invariant distribution on \mathcal{V} such that*

1. *There exists $P \in \mathbb{C}[X]$ such that $P(\partial(\omega))\Theta = 0$*

2. There exists $F \in L_{loc}^1(\mathcal{V})^H$ such that $\Theta = F$ on $\mathcal{V} \cap \mathfrak{q}^{reg}$.

Then $\Theta = F$ as distribution on \mathcal{V} .

We will use the method developed by M. Atiyah in [1]. First we recall some facts about distributions on $\mathbb{R}^r \times \mathbb{R}^m$. Let \mathbb{N} be the set of non-negative integers. For $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$, we set $|\alpha| = \alpha_1 + \dots + \alpha_r$ and

$$x^\alpha = x_1^{\alpha_1} \dots x_r^{\alpha_r}, \quad \partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r}}.$$

For $\varphi \in \mathcal{D}(\mathbb{R}^r \times \mathbb{R}^m)$ and $\varepsilon > 0$, we set $\varphi_\varepsilon(x, y) = \varphi(\frac{x}{\varepsilon}, y)$ for $(x, y) \in \mathbb{R}^r \times \mathbb{R}^m$. For $T \in \mathcal{D}'(\mathbb{R}^r \times \mathbb{R}^m)$ we denote by T_ε the distribution defined by $\langle T_\varepsilon, \varphi \rangle = \langle T, \varphi_\varepsilon \rangle$.

Definition 5.2. Let $V = \{0\} \times \mathbb{R}^m \subset \mathbb{R}^r \times \mathbb{R}^m$ and $T \in \mathcal{D}'(\mathbb{R}^r \times \mathbb{R}^m)$.

1. The distribution T is regular along V if $\lim_{\varepsilon \rightarrow 0} T_\varepsilon = 0$.

2. The distribution T has a degree of singularity along V smaller than k if for all $\alpha \in \mathbb{N}^r$ with $|\alpha| = k$, the distribution $x^\alpha T$ is regular.

We denote by $d_s^\circ T$ the degree of singularity of T along V and we omit in what follows to precise "along V ". Regularity corresponds to a degree of singularity equal to 0.

3. The degree of singularity of T is equal to k if $d_s^\circ T \leq k$ and $d_s^\circ T \not\leq k - 1$.

Lemma 5.3. 1. If $F \in L_{loc}^1(\mathbb{R}^{r+m})$ then $d_s^\circ F = 0$.

2. If $d_s^\circ T = k \geq 1$ then $d_s^\circ(x_i T) = k - 1$ for $i \in \{1, \dots, r\}$.

3. If $d_s^\circ T \leq k$ then $\frac{\partial}{\partial x_i} T \leq k + 1$ for $i \in \{1, \dots, r\}$.

4. Let δ_0 be the Dirac measure at $0 \in \mathbb{R}^r$ and $\delta_0^{(\alpha)} = \partial_x^\alpha \delta_0$. If $S \in \mathcal{D}'(\mathbb{R}^m)$ then the degree of singularity of $\delta_0^{(\alpha)} \otimes S$ is equal to $|\alpha| + 1$.

Proof. 1. Let $F \in L_{loc}^1(\mathbb{R}^{r+m})$ and $\phi \in \mathcal{D}(\mathbb{R}^{r+m})$ with $\text{supp}(\phi) \subset K_1 \times K_2$ where K_1 (resp., K_2) is a compact subset of \mathbb{R}^r (resp., \mathbb{R}^m). One has

$$\left| \int_{\mathbb{R}^r \times \mathbb{R}^m} F(x, y) \phi\left(\frac{x}{\varepsilon}, y\right) dx dy \right| \leq \sup_{(x, y) \in \mathbb{R}^{r+m}} |\phi(x, y)| \int_{(\varepsilon K_1) \times K_2} |F(x, y)| dx dy$$

and the first assertion follows.

2. is clear.

3. Let $\alpha \in \mathbb{N}^r$ such that $|\alpha| = k + 1$. If $\alpha_j \geq 1$ for some $j \in \{1, \dots, r\}$, we set $\bar{\alpha}^j = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots, \alpha_r)$. Let $\varphi \in \mathcal{D}(\mathbb{R}^{r+m})$.

If $\alpha_i \geq 1$, one has

$$\begin{aligned} \langle x^\alpha \frac{\partial}{\partial x_i} T, \varphi_\varepsilon \rangle &= - \langle T, \alpha_i x^{\bar{\alpha}^i} \varphi_\varepsilon + \frac{x^\alpha}{\varepsilon} \left(\frac{\partial}{\partial x_i} \varphi \right)_\varepsilon \rangle \\ &= -\alpha_i \langle x^{\bar{\alpha}^i} T, \varphi_\varepsilon \rangle - \langle x^{\bar{\alpha}^i} T, (x_i \frac{\partial}{\partial x_i} \varphi)_\varepsilon \rangle \end{aligned}$$

thus $(x^\alpha T)_\varepsilon$ converges to 0 since $d_s^\circ T \leq k$.

If $\alpha_i = 0$, we choose j such that $\alpha_j \geq 1$. One has $\langle x^\alpha \frac{\partial}{\partial x_i} T, \varphi_\varepsilon \rangle = - \langle x^{\bar{\alpha}^j} T, (x_j \frac{\partial}{\partial x_i} \varphi)_\varepsilon \rangle$ which tends to 0 as before.

4. We recall that for $i \in \{1, \dots, r\}$, one has

$$x_i^l \delta_0^{(\alpha)} = \begin{cases} (-1)^l \frac{(\alpha_i)!}{(\alpha_i - l)!} \delta_0^{(\alpha_1, \dots, \alpha_i - l, \dots, \alpha_n)} & \text{if } \alpha_i \geq l \\ 0 & \text{if } \alpha_i < l. \end{cases}$$

Hence, one has $x^\alpha \delta_0^{(\alpha)} = (-1)^{|\alpha|} \alpha! \delta_0$ and for all $\beta \in \mathbb{N}^r$ with $|\beta| = |\alpha| + 1$, one has $x^\beta \delta_0^{(\alpha)} = 0$. The assertion follows. \square

Definition 5.4. Let $\Gamma = x^\beta \partial_x^\alpha D$ where D is a differential operator on \mathbb{R}^m . Then Γ increases the degree of singularity at most $|\alpha| - |\beta|$. The integer $|\alpha| - |\beta|$ is called the total degree of Γ in x .

We can define the homogeneous part of highest total degree (in x) of an analytic differential operator developing its coefficients in Taylor series.

Proof of the Theorem. Let $\Theta \in \mathcal{D}'(\mathcal{V})^H$ and $F \in L_{loc}^1(\mathcal{V})^H$ such that $P(\partial(\omega))\Theta = 0$ for a unitary polynomial $P \in \mathbb{C}[X]$ and $\Theta = F$ on $\mathcal{V}^{reg} = \mathcal{V} \cap \mathfrak{q}^{reg}$. We write $\Theta = F + S$ where S is an H -invariant distribution with support contained in $\mathcal{V} - \mathcal{V}^{reg}$. We want to prove that $S = 0$, which is equivalent to $\text{supp}(S) = \emptyset$.

Assuming S is non-zero, we are led to a contradiction. We will study S near an element $Z_0 \in \text{supp}(S)$ chosen as follows:

For $Z_0 \in \text{supp}(S)$ with Jordan decomposition $Z_0 = A_0 + X_0$, we construct the symmetric subpair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ related to A_0 and we set $\mathfrak{q}_{A_0} = \mathfrak{z}^- = \mathfrak{c}^- \oplus \mathfrak{z}_s^-$ as in section 3. Let \mathcal{S}_k be the set of Z_0 in the support of S such that $\text{rank}(\mathfrak{z}_s^-) = k$. Since $\text{supp}(S) \subset \mathcal{V} - \mathcal{V}^{reg}$, if $Z_0 = A_0 + X_0$ belongs to $\text{supp}(S)$ then A_0 is not \mathfrak{q} -regular. One deduces that $\mathcal{S}_0 = \emptyset$. Let $k_0 > 0$ such that $\mathcal{S}_0 = \mathcal{S}_1 = \dots = \mathcal{S}_{k_0-1} = \emptyset$ and $\mathcal{S}_{k_0} \neq \emptyset$.

For $Z_0 = A_0 + X_0$ in \mathcal{S}_{k_0} , we denote by $\mathcal{N}(\mathfrak{z}_s^-) = \mathcal{O}_1 \cup \dots \cup \mathcal{O}_\nu$ the set of nilpotent elements in \mathfrak{z}_s^- as in section 4. Since $\text{supp}(S) \cap (A_0 + \mathcal{N}(\mathfrak{z}_s^-)) \neq \emptyset$, one can choose $j_0 \in \{1, \dots, \nu\}$ such that $\text{supp}(S) \cap (A_0 + \mathcal{O}_i) = \emptyset$ for $i \in \{1, \dots, j_0 - 1\}$ and $\text{supp}(S) \cap (A_0 + \mathcal{O}_{j_0}) \neq \emptyset$.

From now on, we fix $Z_0 = A_0 + X_0$ in \mathcal{S}_{k_0} such that $X_0 \in \mathcal{O}_{j_0}$.

For $\varepsilon > 0$, we denote by \mathcal{W}_ε the set of x in \mathfrak{z}_s^- such that, for any eigenvalue λ of $\text{ad}_{\mathfrak{g}} x$, one has $|\lambda| < \varepsilon$. The choice of k_0 implies that there exists $\varepsilon > 0$ such that $\text{supp}(S) \cap (Z_0 + \mathcal{W}_\varepsilon) \subset \text{supp}(S) \cap (Z_0 + \mathfrak{c}^- + \mathcal{N}(\mathfrak{z}_s^-))$. Hence, we can choose an open neighborhood \mathcal{W}_c of 0 in \mathfrak{c}^- and an open neighborhood \mathcal{W}_s of X_0 in \mathfrak{z}_s^- such that

$$\text{supp}(S) \cap (A_0 + \mathcal{W}_c + \mathcal{W}_s) \subset \text{supp}(S) \cap (A_0 + \mathcal{W}_c + \mathcal{N}(\mathfrak{z}_s^-)). \quad (5.1)$$

First case. $A_0 \notin \mathfrak{c}_{\mathfrak{q}}$ and $X_0 \neq 0$.

We keep the notation of section 4. We fix a normal sl_2 -triple (B_0, Y_0, X_0) in $(\mathfrak{z}_s, \mathfrak{z}_s^+)$. We choose an open neighborhood U_0 of 0 in U , the centralizer of Y_0 in \mathfrak{z}_s^- , as in Lemma 4.4. We keep the notation of this lemma. We recall that the map γ from $H \times^1 \mathfrak{z}^-$ to \mathfrak{q} defined by $\gamma(h, Z) = h \cdot (A_0 + Z)$ is a submersion. Reducing U_0 , \mathcal{W}_c and \mathcal{W}_s if necessary, we may assume

that $\mathcal{W}_c + \Omega_0 \subset \mathcal{W}_c + \mathcal{W}_s \subset \mathfrak{z}^-$ and that $V_0 = \gamma(H \times (\mathcal{W}_c + \Omega_0))$ is an open neighborhood of Z_0 contained in \mathcal{V} .

If T is an H -invariant distribution on \mathcal{V} , we denote by T_0 its restriction to V_0 . By theorem 2.1, one can consider its restriction $T_1 = \text{Res}_{\mathfrak{z}^-} T_0$ to $\mathcal{W}_c + \Omega_0$ with respect to γ . One has $A_0 + \text{supp}(T_1) \subset \text{supp}(T) \cap (A_0 + \mathcal{W}_c + \Omega_0)$.

We set $T_2 = \xi^{1/2} T_1$ where $\xi^{1/2}$ is the analytic function on $\mathcal{W}_c + \Omega_0$ defined in section 3.

Now, we consider the submersion π_0 from $H_s^+ \times U_0 \times \mathcal{W}_c$ to \mathfrak{z}^- defined by $\pi_0(h, X, C) = h \cdot (X_0 + X) + C$. One denotes by T_3 the restriction on $U_0 \times \mathcal{W}_c$ of T_2 with respect to π_0 . We have $X_0 + \text{supp}(T_3) \subset \text{supp}(T_2) \cap (X_0 + U_0)$.

Since F is a locally integrable function, the distribution F_3 is the locally integrable function on $U_0 \times \mathcal{W}_c$ defined by $F_3(X, C) = \xi^{1/2}(C + X)F(C + X)$.

By assumption, the distribution S_3 is non-zero. By (5.1) and Lemma 4.4 (2.), one has $\text{supp}(S_2) = \text{supp}(S_1) \subset \mathcal{W}_c + \Omega_0 \cap \mathcal{N}_{j_0} = \mathcal{W}_c + \mathcal{O}_{j_0}$. We deduce from Lemma 4.4 (3.) that $\text{supp}(S_3) \subset \{0\} \times \mathcal{W}_c$. By ([6], Lemma 3), there exists a family $(S_\alpha)_\alpha$ of $\mathcal{D}'(\mathcal{W}_c)$ such that $S_3 = \sum_{\alpha \in \mathbb{N}^r; |\alpha| \leq l} \delta_0^{(\alpha)} \otimes S_\alpha$ where δ_0 is the Dirac measure at 0 of U_0 and for $\alpha \in \mathbb{N}^r$, the S_α with $|\alpha| = l$ are not all zero.

By assumption, the distribution Θ satisfies $P(\partial(\omega))\Theta = 0$. By Lemma 3.1, one has

$$P\left((\partial(\omega_s) + \partial(\omega_c)) - \mu(Z)\right)\Theta_2 = 0 \text{ on } \mathcal{W}_c + \Omega_0.$$

Using the restriction with respect to π_0 , one obtains

$$P\left(\text{Rad}_U(\partial(\omega_s)) + \partial(\omega_c) - \tilde{\mu}\right)\Theta_3 = 0 \text{ on } U_0 \times \mathcal{W}_c$$

where $\tilde{\mu}(X, C) = \mu(C + X)$ for $X \in U_0$ and $C \in \mathcal{W}_c$.

Let D_0 be the homogeneous part of highest total degree d of $\text{Rad}_U(\partial(\omega_s))$. We set

$$P\left(\text{Rad}_U(\partial(\omega_s)) + \partial(\omega_c) - \tilde{\mu}\right) = D_0^N + D_1$$

where N is the degree of P and D_1 is a differential operator with total degree in X strictly smaller than Nd . Since $\Theta_3 = F_3 + S_3$ with $S_3 = \sum_{\alpha \in \mathbb{N}^r; |\alpha| \leq l} \delta_0^{(\alpha)} \otimes S_\alpha$, we obtain the following relation on $U_0 \times \mathcal{W}_c$:

$$(D_0^N + D_1)S_3 = (D_0^N + D_1)\left(\sum_{\alpha \in \mathbb{N}^r; |\alpha| \leq l} \delta_0^{(\alpha)} \otimes S_\alpha\right) = -(D_0^N + D_1)F_3 \quad (5.2)$$

We study now the degree of singularity along $\{0\} \times \mathcal{W}_c$ of the two members of (5.2).

If X_0 is not a \mathfrak{z}_s^- -distinguished nilpotent element then by Lemma 4.5, the homogeneous part of degree 2 of $\text{Rad}_{U,0}(\partial(\omega_s))$ does not vanish and is a differential operator with constant coefficients of degree 2. Hence the total degree of D_0 is equal to $d = 2$. Since F_3 is a locally integrable function, it follows from Lemma 5.3 that one has $d_s^\circ F_3 = 0$ and $d_s^\circ((D_0^N + D_1)F_3) \leq 2N$. By the same Lemma, one has $d_s^\circ((D_0^N + D_1)S_3) = l + 1 + 2N$. Hence, we have a contradiction.

Assume that X_0 is a \mathfrak{z}_s^- -distinguished nilpotent element. Lemma 4.6 gives $c_0 D_0 = 2x_1 \frac{\partial^2}{\partial x_1^2} + (dim \mathfrak{z}_s^-) \frac{\partial}{\partial x_1} + \sum_{i=2}^r (n_i + 2)x_i \frac{\partial^2}{\partial x_1 \partial x_i} + \sum_{2 \leq i \leq j \leq r} a_{i,j}(X) \frac{\partial^2}{\partial x_j \partial x_i} + \sum_{i=2}^r a_i(X) \frac{\partial}{\partial x_i}$ where $c_0 = \|X_0\|$. Since $a_{i,j}(0) = 0$, the total degree of D_0 is equal to 1.

For $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$, we set $\tilde{\alpha}^i = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \dots, \alpha_r)$ and $\bar{\alpha}^i = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \dots, \alpha_r)$. The relation $x_i \delta_0^{(\alpha)} = -\alpha_i \delta_0^{(\bar{\alpha}^i)}$ and the above expression of D_0 give

$$c_0 D_0 \cdot \delta_0^{(\alpha)} \otimes S_\alpha = \lambda_\alpha \delta_0^{(\tilde{\alpha}^1)} \otimes S_\alpha + \sum_{2 \leq i \leq j \leq r} a_{i,j}(X) \delta^{(\tilde{\alpha}^{i,j})} \otimes S_\alpha + \sum_{i=2}^r a_i(X) \delta^{(\bar{\alpha}^i)} \otimes S_\alpha$$

where

$$\lambda_\alpha = -2(\alpha_1 + 2) + dim \mathfrak{z}_s^- - \sum_{i=2}^r (n_i + 2)(\alpha_i + 1).$$

Since n_1 is equal to 2 and $(\mathfrak{g}, \mathfrak{h})$ is a nice pair, we obtain

$$\lambda_\alpha = -\delta_{\mathfrak{q}}(Z_0) - [2\alpha_1 + \sum_{i=2}^r (n_i + 2)\alpha_i] < 0 \text{ for all } \alpha \in \mathbb{N}^r.$$

Consider $\alpha_0 = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ such that $|\alpha_0| = l$, $S_{\alpha_0} \neq 0$ and α_1 is maximal for these properties. One deduces that the coefficient of $\delta^{(\tilde{\alpha}_0^1)} \otimes S_{\alpha_0}$ in $D_0 \cdot (\sum_{\alpha \in \mathbb{N}^r; |\alpha|=l} \delta_0^{(\alpha)} \otimes S_\alpha)$ is non-zero. Thus, the degree of singularity of $(D_0^N + D_1)S_3$ is equal to $1 + l + N$. Since F_3 is locally integrable and the total degree of D_0 is equal to 1, we have $d_s^c(D_0^N + D_1)F_3 \leq N$. This gives a contradiction in (5.2).

Second case. $A_0 \in \mathfrak{c}_{\mathfrak{q}}$ and $X_0 \neq 0$.

The symmetric pair $(\mathfrak{z}_s, \mathfrak{z}_s^+)$ is equal to $(\mathfrak{g}_s, \mathfrak{h}_s)$. We just consider the submersion π_0 from $H \times U_0 \times \mathcal{W}_c$ to \mathfrak{q} defined by $\pi_0(h, X, C) = h \cdot (X_0 + X) + A_0 + C$ where U_0 is defined as in Lemma 4.4 for the symmetric pair $(\mathfrak{g}_s, \mathfrak{h}_s)$.

For $T \in \mathcal{D}'(\mathfrak{q})^H$, we denote by T_1 the restriction of T to $U_0 \times \mathcal{W}_c$ with respect to π_0 . As in the first case, we have $\Theta_1 = F_1 + S_1$ where F_1 is a locally integrable function on $U_0 \times \mathcal{W}_c$ and S_1 is a non-zero distribution such that $\text{supp}(S_1) \subset \{0\} \times \mathcal{W}_c$. Moreover the distribution Θ_1 satisfies the relation

$$P(\mathcal{R}ad_U(\partial(\omega_s)) + \partial(\omega_c))\Theta_1 = 0 \text{ on } U_0 \times \mathcal{W}_c.$$

The same arguments as in the first case lead to the contradiction $S_1 = 0$.

Third case. $X_0 = 0$.

The open sets \mathcal{W}_c and \mathcal{W}_s satisfy $\text{supp}(S) \cap (A_0 + \mathcal{W}_c + \mathcal{W}_s) \subset \text{supp}(S) \cap (A_0 + \mathcal{W}_c + \mathcal{N}(\mathfrak{z}_s^-))$. By the choice of j_0 , we deduce that $\text{supp}(S) \cap (A_0 + \mathcal{W}_c + \mathcal{W}_s) \subset \text{supp}(S) \cap (A_0 + \mathcal{W}_c)$.

If $A_0 \in \mathfrak{c}_{\mathfrak{q}}$, then $V_0 = A_0 + \mathcal{W}_c + \mathcal{W}_s$ is an open neighborhood of A_0 in \mathfrak{q} . We identify \mathfrak{q} with $\mathfrak{q}_s \times \mathfrak{c}_{\mathfrak{q}}$. Thus, the restriction S_0 of S to V_0 is different from zero and satisfies $\text{supp}(S_0) \subset \{0\} \times (A_0 + \mathcal{W}_c)$. On the other hand, one has $P(\partial(\omega))S_0 = -P(\partial(\omega))F_1|_{V_0}$. Since $\partial(\omega)$ is a second order operator with constant coefficients, we obtain a contradiction as above.

If $A_0 \notin \mathfrak{c}_{\mathfrak{q}}$, we may assume that $\mathcal{W}_c + \mathcal{W}_s \subset \mathfrak{z}^-$. We denote by T_1 the restriction of an H -invariant distribution T to $\mathcal{W}_c + \mathcal{W}_s$ with respect to the submersion γ from $H \times \mathfrak{z}^-$ to \mathfrak{q} and we consider $T_2 = \xi^{1/2} T_1$ as distribution on $\mathcal{W}_s \times \mathcal{W}_c$. Thus, we have $S_2 \neq 0$ and $\text{supp}(S_2) = \{0\} \times \mathcal{W}_c$. Moreover, the distribution $\Theta_2 = F_2 + S_2$ satisfies $P((\partial(\omega_s) + \partial(\omega_c)) - \mu(Z))\Theta_2 = 0$ on $\mathcal{W}_s \times \mathcal{W}_s$ by Lemma 3.1. This is equivalent to

$$P((\partial(\omega_s) + \partial(\omega_c)) - \mu(Z))S_2 = -P((\partial(\omega_s) + \partial(\omega_c)) - \mu(Z))F_2.$$

Since $\partial(\omega_s)$ is a second order operator with constant coefficients, we obtain a contradiction as above.

This achieves the proof of the Theorem. \square

6 Application to $(\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))$

On $G = GL(4, \mathbb{R})$ and its Lie algebra $\mathfrak{g} = \mathfrak{gl}(4, \mathbb{R})$, we consider the involution σ defined by $\sigma(X) = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} X \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ where I_2 is the 2×2 identity matrix. We have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ with

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}; A, B \in \mathfrak{gl}(2, \mathbb{R}) \right\} \text{ and } \mathfrak{q} = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix}; Y, Z \in \mathfrak{gl}(2, \mathbb{R}) \right\}.$$

By ([7] Theorem 6.3), the symmetric pair $(\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))$ is a nice pair.

We first recall some results of [3]. Let $\kappa(X, X') = \frac{1}{2} \text{tr}(XX')$. The restriction of κ to the derived algebra of \mathfrak{g} is a multiple of the Killing form. Let $S(\mathfrak{q}_{\mathbb{C}})^{H_{\mathbb{C}}}$ be subalgebra of $S(\mathfrak{q}_{\mathbb{C}})$ of all elements invariant under $H_{\mathbb{C}}$. We identify $S(\mathfrak{q}_{\mathbb{C}})^{H_{\mathbb{C}}}$ with the algebra of $H_{\mathbb{C}}$ -invariant differential operators on $\mathfrak{q}_{\mathbb{C}}$ with constant coefficients. Using κ , we identify $S(\mathfrak{q}_{\mathbb{C}})^{H_{\mathbb{C}}}$ with the algebra $\mathbb{C}[\mathfrak{q}_{\mathbb{C}}]^{H_{\mathbb{C}}}$ of $H_{\mathbb{C}}$ -invariant polynomials on $\mathfrak{q}_{\mathbb{C}}$. A basis of $\mathbb{C}[\mathfrak{q}_{\mathbb{C}}]^{H_{\mathbb{C}}}$ is given by $Q(X) = \frac{1}{2} \text{tr}(X^2)$ and $S(X) = \det(X)$. The Casimir polynomial is just a multiple of Q .

By ([3] Lemma 1.3.1), the H -orbit of a semisimple element $X = \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix}$ of \mathfrak{q} is characterized by $(Q(X), S(X))$ or by the set $\{\nu_1(X), \nu_2(X)\}$ of eigenvalues of YZ , where the functions ν_1 and ν_2 are defined as follows: let Y be the Heaviside function. Let $S_0 = Q^2 - 4S$ and $\delta = \iota^{Y(-S_0)} \sqrt{|S_0|}$. We set

$$\nu_1 = (Q + \delta)/2 \quad \text{and} \quad \nu_2 = (Q - \delta)/2.$$

Regular elements of \mathfrak{q} are semisimple elements with 2 by 2 distinct eigenvalues or equivalently, semisimple elements X of \mathfrak{q} such that $\nu_1(X)\nu_2(X)(\nu_1(X) - \nu_2(X)) \neq 0$ ([3] Remarque 1.3.1).

Let χ be the character of $\mathbb{C}[\mathfrak{q}_{\mathbb{C}}]^{H_{\mathbb{C}}}$ defined by $\chi(Q) = \lambda_1 + \lambda_2$ and $\chi(S) = \lambda_1 \lambda_2$ where λ_1 and λ_2 are two complex numbers satisfying $\lambda_1 \lambda_2 (\lambda_1 - \lambda_2) \neq 0$.

For an open H -invariant subset \mathcal{V} in \mathfrak{q} , we denote by $\mathcal{D}'(\mathcal{V})_{\chi}^H$ the set of H -invariant distributions T with support in \mathcal{V} such that $\partial(P)T = \chi(P)T$ for all $P \in \mathbb{C}[\mathfrak{q}_{\mathbb{C}}]^{H_{\mathbb{C}}}$. Let \mathcal{N} be the set of nilpotent elements of \mathfrak{q} and $\mathcal{U} = \mathfrak{q} - \mathcal{N}$ its complement. In [3], we describe a basis of

the subspace of $\mathcal{D}'(\mathcal{U})_\chi^H$ consisting of locally integrable functions. More precisely, we obtain the following result.

We consider the Bessel operator $L_c = 4 \left(z \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} \right)$ on \mathbb{C} and its analogous $L = 4 \left(t \frac{d^2}{dt^2} + \frac{d}{dt} \right)$ on \mathbb{R} . Let $\mathcal{Sol}(L_c, \lambda)$ (resp., $\mathcal{Sol}(L, \lambda)$) be the set of holomorphic (resp., real analytic) functions f on $\mathbb{C} - \mathbb{R}_-$ (resp., \mathbb{R}^*) such that $L_c f = \lambda f$ (resp., $L f = \lambda f$). For $\lambda \in \mathbb{C}^*$, we set

$$\Phi_\lambda(z) = \sum_{n \geq 0} \frac{(\lambda z)^n}{4^n (n!)^2} \quad \text{and} \quad w_\lambda(z) = \sum_{n \geq 0} \frac{a(n)(\lambda z)^n}{4^n (n!)^2},$$

where $a(x) = -2 \frac{\Gamma'(x+1)}{\Gamma(x+1)}$. Then $(\Phi_\lambda, W_\lambda = w_\lambda + \log(\cdot) \Phi_\lambda)$ form a basis of $\mathcal{Sol}(L_c, \lambda)$, where \log is the principal determination of the logarithm function on $\mathbb{C} - \mathbb{R}_-$ and $(\Phi_\lambda, W_\lambda^r = w_\lambda + \log|\cdot| \Phi_\lambda)$ form a basis of $\mathcal{Sol}(L, \lambda)$.

For two functions f and g defined over \mathbb{C} , we set

$$S^+(f, g)(X) = f(\nu_1(X))g(\nu_2(X)) + f(\nu_2(X))g(\nu_1(X))$$

and

$$[f, g](X) = f(\nu_1(X))g(\nu_2(X)) - f(\nu_2(X))g(\nu_1(X)).$$

We define the following functions on \mathfrak{q}^{reg} :

1.

$$F_{ana} = \frac{[\Phi_{\lambda_1}, \Phi_{\lambda_2}]}{\nu_1 - \nu_2}$$

2.

$$F_{sing} = \frac{[\Phi_{\lambda_1}, w_{\lambda_2}] + [w_{\lambda_1}, \Phi_{\lambda_2}] + \log|\nu_1 \nu_2| [\Phi_{\lambda_1}, \Phi_{\lambda_2}]}{\nu_1 - \nu_2}$$

3. For $(A, B) \in \{(\Phi_{\lambda_1}, \Phi_{\lambda_2}), (\Phi_{\lambda_1}, W_{\lambda_2}^r), (W_{\lambda_1}^r, \Phi_{\lambda_2}), (W_{\lambda_1}^r, W_{\lambda_2}^r)\}$, we set

$$F_{A,B}^+ = Y(S_0) \frac{S^+(A, B)}{|\nu_1 - \nu_2|}$$

where $S_0 = Q^2 - 4S \in \mathbb{C}[\mathfrak{q}_{\mathbb{C}}]^{H_{\mathbb{C}}}$ and Y is the Heveaside function.

Theorem 6.1. ([3] Theorem 5.2.2 and Corollary 5.3.1).

1. The functions F_{ana} and F_{sing} are locally integrable on \mathfrak{q} .

2. For $(A, B) \in \{(\Phi_{\lambda_1}, \Phi_{\lambda_2}), (\Phi_{\lambda_1}, W_{\lambda_2}^r), (W_{\lambda_1}^r, \Phi_{\lambda_2}), (W_{\lambda_1}^r, W_{\lambda_2}^r)\}$, the functions $F_{A,B}^+$ are locally integrable on \mathcal{U} .

3. The family F_{ana}, F_{sing} and $F_{A,B}^+$, with (A, B) as above form a basis \mathcal{B} of the subspace of $\mathcal{D}'(\mathcal{U})_\chi^H$ consisting of distributions given by a locally integrable function.

Corollary 6.2. Any invariant distribution of $\mathcal{D}'(\mathcal{U})_\chi^H$ is given by a locally integrable function on \mathcal{U} . In particular, the family \mathcal{B} defined in the previous Theorem is a basis of $\mathcal{D}'(\mathcal{U})_\chi^H$.

Proof. Let $T \in \mathcal{D}'(\mathcal{U})_\chi^H$. We denote by F its restriction to \mathcal{U}^{reg} . By ([7] Theorem 5.3 (i)), F is an analytic function on \mathcal{U}^{reg} satisfying $(*) \quad \partial(P)F = \chi(P)F$ on \mathcal{U}^{reg} for all $P \in \mathbb{C}[\mathfrak{q}_\mathbb{C}]^{H^c}$.

In ([3] section 4.), we describe the analytic solutions of $(*)$ in terms of Φ_λ , W_λ and W_λ^r for $\lambda = \lambda_1$ and λ_2 . By the asymptotic behaviour of orbital integrals near non-zero semisimple elements ([3] Theorems 3.3.1 and 3.4.1), and the Weyl integration formula ([3] Lemma 3.1.2), one deduces that $F \in L_{loc}^1(\mathcal{U})^H$. Theorem 5.1 gives the result. \square

Corollary 6.3. *Any invariant distribution of $\mathcal{D}'(\mathfrak{q})_\chi^H$ is given by a locally integrable function on \mathfrak{q} .*

Proof. Let $T \in \mathcal{D}'(\mathfrak{q})_\chi^H$. By Corollary 6.2, the restriction of T to \mathcal{U} is a linear combination of elements of \mathcal{B} . By Theorem 5.1 and Theorem 6.1, it is enough to prove that the functions $F_{A,B}^+$, with $(A, B) \in \{(\Phi_{\lambda_1}, \Phi_{\lambda_2}), (\Phi_{\lambda_1}, W_{\lambda_2}^r), (W_{\lambda_1}^r, \Phi_{\lambda_2}), (W_{\lambda_1}^r, W_{\lambda_2}^r)\}$ are locally integrable on \mathfrak{q} or equivalently, that the integral $\int_{\mathfrak{q}} |F_{A,B}^+(X)f(X)|dX$ is finite for all positive function $f \in \mathcal{D}(\mathfrak{q})$. For this, we will use the Weyl integration formula ([5] Proposition 1.8 and Theorem 1.27).

For $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_j = \pm$, we define

$$\mathfrak{a}_\varepsilon = \left\{ X_\varepsilon(u_1, u_2) = \left(\begin{array}{cc|cc} 0 & & u_1 & 0 \\ & & 0 & u_2 \\ \hline \varepsilon_1 u_1 & 0 & & \\ 0 & \varepsilon_2 u_2 & & 0 \end{array} \right); (u_1, u_2) \in \mathbb{R}^2 \right\}.$$

and

$$\mathfrak{a}_2 = \left\{ \left(\begin{array}{cc|cc} 0 & & \tau & -\theta \\ & & \theta & \tau \\ \hline \tau & -\theta & & \\ \theta & \tau & & 0 \end{array} \right); (\theta, \tau) \in \mathbb{R}^2 \right\}$$

By ([3], Lemma 1.2.1), the subspaces $\mathfrak{a}_{++}, \mathfrak{a}_{+-}, \mathfrak{a}_{--}$ and \mathfrak{a}_2 form a system of representatives of H -conjugaison classes of Cartan subspaces in \mathfrak{q} . By ([3] Remark 1.3.1), an element $X \in \mathfrak{q}$ satisfies $S_0(X) \geq 0$ if and only if X is H -conjugate to an element of \mathfrak{a}_ε for some ε . Furthermore, one has $\{\nu_1(X_\varepsilon(u_1, u_2)), \nu_2(X_\varepsilon(u_1, u_2))\} = \{\varepsilon_1 u_1^2, \varepsilon_2 u_2^2\}$.

Let f be a positive function in $\mathcal{D}(\mathfrak{q})$. We define the orbital integral of f on \mathfrak{q}^{reg} by

$$\mathcal{M}(f)(X) = |\nu_1(X) - \nu_2(X)| \int_{H/Z_H(X)} f(h.X) dX$$

where $Z_H(X)$ is the centralizer of X in H and dh is an invariant measure on $H/Z_H(X)$.

By ([5] Theorem 1.23), the orbital integral $\mathcal{M}(f)$ is a smooth function on \mathfrak{q}^{reg} and there exists a compact subset Ω of \mathfrak{q} such that $\mathcal{M}(f)(X) = 0$ for all regular element X in the complement of Ω .

Since $F_{A,B}^+$ is zero on \mathfrak{a}_2^{reg} , one deduces from the Weyl integration formula that there exist positive constants C_ε (only depending of the choice of measures), such that one has

$$\begin{aligned} \int_{\mathfrak{q}} F_{A,B}^+(X) f(X) dX &= \sum_{\varepsilon \in \{(++), (+-), (--)\}} C_{\varepsilon} \int_{\mathbb{R}^2} F_{A,B}^+(X_{\varepsilon}(u_1, u_2)) \\ &\quad \times \mathcal{M}(f)(X_{\varepsilon}(u_1, u_2)) |u_1 u_2 (\varepsilon_1 u_1^2 - \varepsilon_2 u_2^2)| du_1 du_2. \end{aligned}$$

By definition of $F_{A,B}^+$, there exist positive constants C, C_1 and C_2 such that, for all $X_{\varepsilon}(u_1, u_2) \in \Omega^{reg}$, one has

$$|(\varepsilon_1 u_1^2 - \varepsilon_2 u_2^2) F_{A,B}^+(X_{\varepsilon}(u_1, u_2))| \leq C(C_1 + |\log |u_1||)(C_2 + |\log |u_2||).$$

One deduces easily the corollary from the following Lemma. □

Lemma 6.4. *Let $f \in \mathcal{D}(\mathfrak{q})$. Then there exist positive constants C', C'_1, C'_2 such that, for all $X_{\varepsilon}(u_1, u_2) \in \mathfrak{q}^{reg}$ one has*

$$|\mathcal{M}(f)(X_{\varepsilon}(u_1, u_2))| \leq C'(C'_1 + |\log |u_1||)(C'_2 + |\log |u_2||).$$

Proof. Let $H = KNA$ be the Iwasawa decomposition of H with $K = O(2) \times O(2)$, $N = N_0 \times N_0$ where N_0 consists of 2 by 2 unipotent upper triangular matrices and A is the set of diagonal matrices in H . It is easy to see that the centralizer of X in H is the set of diagonal matrices $diag((\alpha, \beta, \alpha, \beta)$ with $(\alpha, \beta) \in (\mathbb{R}^*)^2$. Hence $H/Z_H(X)$ is isomorphic to $K \times N \times \{diag(e^x, e^y, 1, 1); x, y \in \mathbb{R}\}$.

For $\xi \in \mathbb{R}$, we set $n_{\xi} = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$. We define the function \tilde{f} by $\tilde{f}(X) = \int_K f(k \cdot X) dk$. Then, one has

$$\mathcal{M}(f)(X_{\varepsilon}(u_1, u_2)) = |\varepsilon_1 u_1^2 - \varepsilon_2 u_2^2| \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \tilde{f}(Y(u, \varepsilon, x, y, \xi, \eta)) d\xi d\eta \right) dx dy$$

with

$$Y(u, \varepsilon, x, y, \xi, \eta) = \left(\begin{pmatrix} n_{\xi} & 0 \\ 0 & n_{\eta} \end{pmatrix} diag(e^x, e^y, 1, 1) \right) \cdot X_{\varepsilon, u}.$$

Writing $Y(u, \varepsilon, x, y, \xi, \eta) = \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix}$, one has

$$Y = \begin{pmatrix} u_1 e^x & -\eta u_1 e^x + e^y \xi u_2 \\ 0 & u_2 e^y \end{pmatrix} \text{ and } Z = \begin{pmatrix} \varepsilon_1 u_1 e^{-x} & -\xi \varepsilon_1 u_1 e^{-x} + \eta \varepsilon_2 u_2 e^{-y} \\ 0 & \varepsilon_2 u_2 e^{-y} \end{pmatrix}.$$

Since $f \in \mathcal{D}(\mathfrak{q})$, the function \tilde{f} has compact support in \mathfrak{q} . Identify \mathfrak{q} with \mathbb{R}^8 , there exists $T > 0$ such that $\text{supp}(\tilde{f}) \subset [-T, T]^8$. If $\tilde{f}(Y(u, \varepsilon, x, y, \xi, \eta)) \neq 0$ then we have the following inequalities:

1. $|u_1 e^{\pm x}| \leq T$ and $|u_2 e^{\pm y}| \leq T$,
2. $|\eta u_1 e^x + e^y \xi u_2| \leq T$,
3. $|\xi \varepsilon_1 u_1 e^{-x} + \eta \varepsilon_2 u_2 e^{-y}| \leq T$.

Changing the variables (ξ, η) in $(r, s) = (\xi u_2 e^y - \eta u_1 e^x, -\xi \varepsilon_1 u_1 e^{-x} + \eta \varepsilon_2 u_2 e^{-y})$, we obtain the result. \square

Remark. By ([3] Corollary 5.3.1), the function F_{ana} defines an invariant eigendistribution on \mathfrak{q} . At this stage, we don't know if it is the case for the functions F_{sing} and $F_{A,B}^+$. Indeed, the proof of Theorem 6.1 of [3] is based on integration by parts using estimates of orbital integrals and some of their derivatives near non-zero semisimple elements of \mathfrak{q} . To determine if F_{sing} and $F_{A,B}^+$ are eigendistributions using the same method, we have to know the behavior of derivatives of orbital integrals near 0.

References

- [1] M. Atiyah, *Characters of semi-simple Lie groups*, (lecture given in Oxford), Mathematical Institute, Oxford, (1976).
- [2] E. Galina, Y. Laurent, *\mathcal{D} -modules and characters of semisimple Lie groups*, Duke Math. J. Volume 123, Number 2 (2004), 265-309.
- [3] P. Harinck et N. Jacquet, Distributions propres invariantes sur la paire symétrique $(\mathfrak{gl}(4, \mathbb{R}), \mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(2, \mathbb{R}))$, Journal of Funct. Anal. 261 (2011), 2362–2436.
- [4] Harish-Chandra, *Invariant distributions on Lie Algebras*, Am. J. Math. 86, (1964), 271–309.
- [5] J. Orloff, *Orbital integrals on symmetric spaces*, Noncommutative harmonic analysis and Lie groups" (Marseille-Luminy, 1985), Lecture Notes in Math., Vol 1243, Springer, Berlin, 1987, p. 198-239.
- [6] L. Schwartz, *Thorie des distributions*, Hermann, (1950).
- [7] J. Sekiguchi, *Invariant spherical hyperfunctions on the tangent space of a symmetric space*, Algebraic groups and related topics, Advanced Studies in Pure Mathematics, vol. 6, (1985), 83–126.
- [8] G. Van Dijk, *Invariant eigendistributions on the tangent space of a rank one semisimple symmetric space I*, Math. Ann., Vol. 268, (1984), p. 405–416.
- [9] G. Van Dijk, *Orbits on real affine symmetric spaces*, I. Proc. Kon. Ned. Ak. Wet. A86, (1983) 51-66.
- [10] V. S. Varadarajan, *Harmonic Analysis on Real Reductive Groups*, Lecture Notes in Mathematics, Vol 576, Springer-Verlag, Berlin/Heidelberg/New York, 1977.